

Deforming the Window of a Gabor Frame: the Ellipsoid Method

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Abstract

In a recent paper in *Appl. Comput. Harmon. Anal.* **38**(2), 196–221 (2014) we have introduced and studied the notion of weak Hamiltonian deformation of a Gabor (=Weyl–Heisenberg) frame. In this Note we use these results to prove that one can modify the window of a Gabor frame using certain metaplectic operators provided that one modifies only a finite number of points of the frame lattice.

1 Introduction

In a previous work [5] we initiated the study of weak Hamiltonian deformations of Gabor frames using ideas borrowed from semiclassical mechanics. In this Note we show that this method allows to modify the window of a Gabor frame using families of metaplectic operators while only replacing a finite number of lattice points with new points contained in an arbitrarily chosen phase space (or time-frequency) ellipsoid. As in [5] we define Gabor frames as follows: let ϕ be a non-zero square integrable function on \mathbb{R}^n (hereafter called the *window*) and Λ a discrete subset of \mathbb{R}^{2n} (the *lattice*). The corresponding *Gabor*, or *Weyl–Heisenberg*, system is the set

$$\mathcal{G}^h(\phi, \Lambda) = \{\widehat{T}(z)\phi : z \in \Lambda\} \quad (1)$$

where $\widehat{T}(z) = e^{-i\sigma(\hat{z}, z)/\hbar}$ is the Heisenberg operator. $\mathcal{G}^h(\phi, \Lambda)$ is called a \hbar -frame for $L^2(\mathbb{R}^n)$ if there exist constants $A, B > 0$ (the *frame bounds*) such that

$$A\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\widehat{T}(z)\phi)|^2 \leq B\|\psi\|^2 \quad (2)$$

for every $\psi \in L^2(\mathbb{R}^n)$. When the parameter \hbar is chosen equal to $1/2\pi$ the notion of \hbar -frame coincides with the usual “naive” notion of Gabor frame [7]. The advantage of using definition (2) is that it highlights the symplectic covariance of Gabor frames: if $S \in \text{Sp}(n)$ is a symplectic automorphism of \mathbb{R}^{2n} then $\mathcal{G}^h(\widehat{S}\phi, S\Lambda)$ is a \hbar -frame if and only if it is the case for $\mathcal{G}^h(\phi, \Lambda)$ (\widehat{S} is anyone of the two metaplectic operators corresponding to S).

In this Note we prove a simple consequence of the weak Hamiltonian deformation theory; we call it the “ellipsoid method” because it relies on the fact that a phase space ellipsoid may be viewed as the energy hypersurface of a Hamilton function that is a quadratic form in the x_j, p_k variables. We will see in a forthcoming work that this method can be extended to arbitrary Hamiltonian functions.

2 Weak Hamiltonian Deformations

Let $H \in C^2(\mathbb{R}^{2n})$; we denote by (f_t^H) the Hamiltonian function H . In [5] we proved the following covariance result for \hbar -frames:

Theorem 1 *Let $\Lambda_t = f_t^H(\Lambda)$, $z_t = f_t^H(z_0)$, $\widehat{S}_t(z_0) = Df_t^H(z_0)$ (z_0 an arbitrary point in \mathbb{R}^{2n}). For $\phi \in L^2(\mathbb{R}^n)$ set*

$$\phi_t = \widehat{T}(z_t)\widehat{S}_t(z_0)\widehat{T}(z_0)^{-1}\phi. \quad (3)$$

The Gabor system $\mathcal{G}^h(\phi_t, \Lambda_t)$ is a \hbar -frame if and only if $\mathcal{G}^h(\phi, \Lambda)$ is a \hbar -frame; when this the case both frames have the same bounds.

The proof of this result is based on the commutation and addition properties of the Heisenberg operators, and on their symplectic covariance [3, 4]. Formula (3) corresponds (up to an unimportant phase factor) to the semi-classical propagation [8, 12] of a wavepacket centered at the point z_0 .

Theorem 1 has been recently applied [6] to the study of the stability of Gabor frames under small time Hamiltonian evolutions.

3 Main Result: Precise Statement and Proof

We assume from now on that the lattice Λ is “ δ -separated”: there exists $\delta > 0$ such that $|z - z'| > \delta$ for all $(z, z') \in \Lambda \times \Lambda$ such that $z \neq z'$. Under this assumption we have:

Lemma 2 *Let Σ be a closed hypersurface in phase space \mathbb{R}^{2n} and let*

$$\Sigma_\varepsilon = \bigcup_{z \in \Sigma} \overline{B}^{2n}(z, \varepsilon)$$

be the “ ε -thickening” of Σ ($\overline{B}^{2n}(z, \varepsilon)$ is the closed ball with radius ε centered at z). There exists ε such that $\Sigma_\varepsilon \cap \Lambda = \Sigma \cap \Lambda$.

This Lemma means that if ε is chosen small enough, the ε -thickening of Σ will contain no more points of the lattice than those which are already on the hypersurface Σ .

We note that surprisingly enough, even in the simple case where Σ is an ellipsoid as will be supposed below, the question of determining how many points of the lattice belong to Σ or its interior is in general very difficult; it harks back to an early result of Landau [10, 11] and has unexpected connections with number theory. We recall that a closed hypersurface is orientable, and hence separates \mathbb{R}^{2n} in two connected components, one of them (“the interior”) being bounded (Jordan–Brouwer theorem).

Let us view the quadratic function $H(z) = \frac{1}{2}Mz \cdot z$ (M positive definite) as a Hamiltonian function. The corresponding Hamilton equations of motion are [1, 3, 4] $\dot{z}(t) = JMz(t)$ hence their solution is given by the simple formula $z(t) = e^{tJM}z(0)$. As is customary, we call the mapping S_t taking the initial value $z(0)$ to the solution $z(t)$ at time t the “Hamiltonian flow” determined by H . We thus have $z(t) = S_t z(0)$ and we may identify S_t with the matrix e^{tJM} , which is symplectic. We thus have $S_t \in \text{Sp}(n)$ for every $t \in \mathbb{R}$. It now follows from general principles (the path lifting theorem, see [3] and the references therein) that when t varies, S_t describes a curve in $\text{Sp}(n)$ passing through the identity at time $t = 0$. This curve can be lifted to a unique curve $t \mapsto \hat{S}_t$ in $\text{Mp}(n)$ passing through the identity at time $t = 0$. Here $\text{Mp}(n)$ is the metaplectic group (it is a unitary representation of the double covering group of $\text{Sp}(n)$; for a detailed study of $\text{Mp}(n)$ see [2, 3, 4]).

Theorem 3 *Let $\mathcal{G}^h(\phi, \Lambda)$ be a Gabor system. Consider the ellipsoid*

$$\Sigma = \{z \in \mathbb{R}^n : \frac{1}{2}Mz \cdot z = E\}$$

and set $F = \Omega \cap \Lambda = \emptyset$ (Ω the compact set bounded by Σ). Let $S_t = e^{tJM} \in \text{Sp}(n)$ and let $\hat{S}_t \in \text{Mp}(n)$ be obtained as described above. Set $F_t = S_t(F)$. Then $\mathcal{G}^h(\hat{S}_t\phi, \Lambda' \cup F_t)$ is a \hbar -frame if and only if $\mathcal{G}^h(\phi, \Lambda)$ is \hbar -frame (with same frame bounds). In particular, if $F = \emptyset$ then $\mathcal{G}^h(\hat{S}_t\phi, \Lambda)$ is a \hbar -frame if and only if $\mathcal{G}^h(\phi, \Lambda)$ is a \hbar -frame.

Proof. That $S_t \in \mathrm{Sp}(n)$ is clear since $JM \in \mathfrak{sp}(n)$ (the symplectic Lie algebra). Let Ω be the compact set bounded by Σ . Choosing ε such that $\Sigma_\varepsilon \cap \Lambda = \Sigma \cap \Lambda$ as in Lemma 2, let χ_ε be a smooth function on \mathbb{R}^{2n} such that

$$\chi_\varepsilon(z) = \begin{cases} 1 & \text{if } z \in \Omega \cup \Sigma_{\varepsilon/2} \\ 0 & \text{if } z \notin \Omega \cup \Sigma_\varepsilon \end{cases} \quad (4)$$

(the existence of χ_ε follows from Tietze–Urysohn’s lemma). Setting $H^\varepsilon = H\chi_\varepsilon$ the support of H^ε is $\Omega \cup \Sigma_\varepsilon$ and we have $H^\varepsilon(z) = H(z)$ for $z \in \Omega \cup \Sigma_{\varepsilon/2}$. The Hamiltonian flow (f_t^ε) determined by H^ε thus satisfies

$$f_t^\varepsilon(z) = \begin{cases} S_t z & \text{if } z \in \Omega \cup \Sigma_{\varepsilon/2} \\ z & \text{if } z \notin \Omega \cup \Sigma_\varepsilon \end{cases}. \quad (5)$$

Notice that if $z \in \Sigma$ then $S_t z \in \Sigma$ for all $t \in \mathbb{R}$ since Σ is an “energy hypersurface” for H . The result now follows from Theorem 1 taking $z_0 = 0$ since $z_t = 0$ and $S_t(0) = S_t$. ■

It is easy to prove a converse to Theorem 3: given a symplectic matrix $S = e^X$ ($X \in \mathfrak{sp}(n)$, the symplectic Lie algebra) we can associate a symplectic flow $S_t = e^{tX}$, namely that of the ellipsoid Σ with $JM = X$. The case of an arbitrary $S \in \mathrm{Sp}(n)$ is slightly more complicated since it leads to time-dependent Hamiltonians: for an arbitrary symplectic matrix S it is not true in general that $S = e^X$ for some $X \in \mathfrak{sp}(n)$; however one can show [9, 13] that there exists a C^1 path $t \mapsto S_t$ of symplectic matrices such that $S_0 = I$ and $S_1 = S$. To this (not uniquely defined) path corresponds a generally time-dependent quadratic Hamiltonian H such that $f_t^H = S_t$. This case will be treated in detail in a forthcoming publication.

4 Discussion

Theorem 3 is a straightforward consequence of Theorem 1 obtained by truncating the initial Hamiltonian: the generalization of Theorem 3 to arbitrary Hamiltonian functions can be made along the same lines. All these results are easily extended (under certain supplementary assumptions) to the case where the window ϕ belongs to a modulation space (cf. [5, 6]).

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